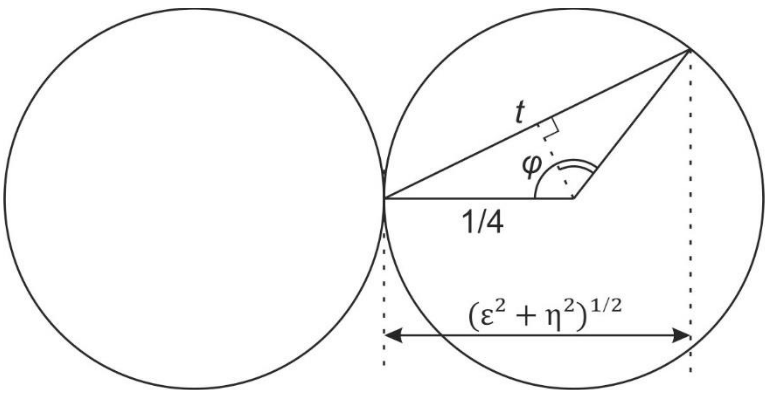
Däumler's mapping onto the horn torus

To visualize the mapping to the surface of the horn torus using Mathcad software, I need to specify parametric equations, in which two parameters x and y determine the three Cartesian coordinates ξ, η and ζ of any point on the surface:

$$F(x,y) = F(\xi(x,y), \eta(x,y), \zeta(x,y))$$

For this, I need to derive formulas for $\xi(x,y)$, $\eta(x,y)$ and $\zeta(x,y)$. Let's consider an auxiliary drawing (below).



It's obvious that $\xi = x \cdot \frac{\sqrt{\xi^2+\eta^2}}{\sqrt{x^2+y^2}}$ and $\eta = y \cdot \frac{\sqrt{\xi^2+\eta^2}}{\sqrt{x^2+y^2}}$, where:

$$\sqrt{\xi^2+\eta^2} = t \cdot \cos\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = t \cdot \sin\frac{\phi}{2}$$

and

$$t = 2 \cdot \frac{1}{4}\sin\frac{\phi}{2} = \frac{1}{2}\sin\frac{\phi}{2}$$

Furthermore:

$$\zeta = t \cdot \sin\left(\frac{\pi}{2} - \frac{\phi}{2}\right) + \frac{1}{2} = t \cdot \cos\frac{\phi}{2} + \frac{1}{2} = \frac{1}{2}\sin\frac{\phi}{2} \cdot \cos\frac{\phi}{2} + \frac{1}{2} = \frac{1}{4}\sin\phi + \frac{1}{2}$$

The Däumler's formula for the "meridian" angle ϕ is:

$$\phi = 2 \cdot \operatorname{arccot}(-\,ln(|\tan(\operatorname{arccot}(|z|))|))$$

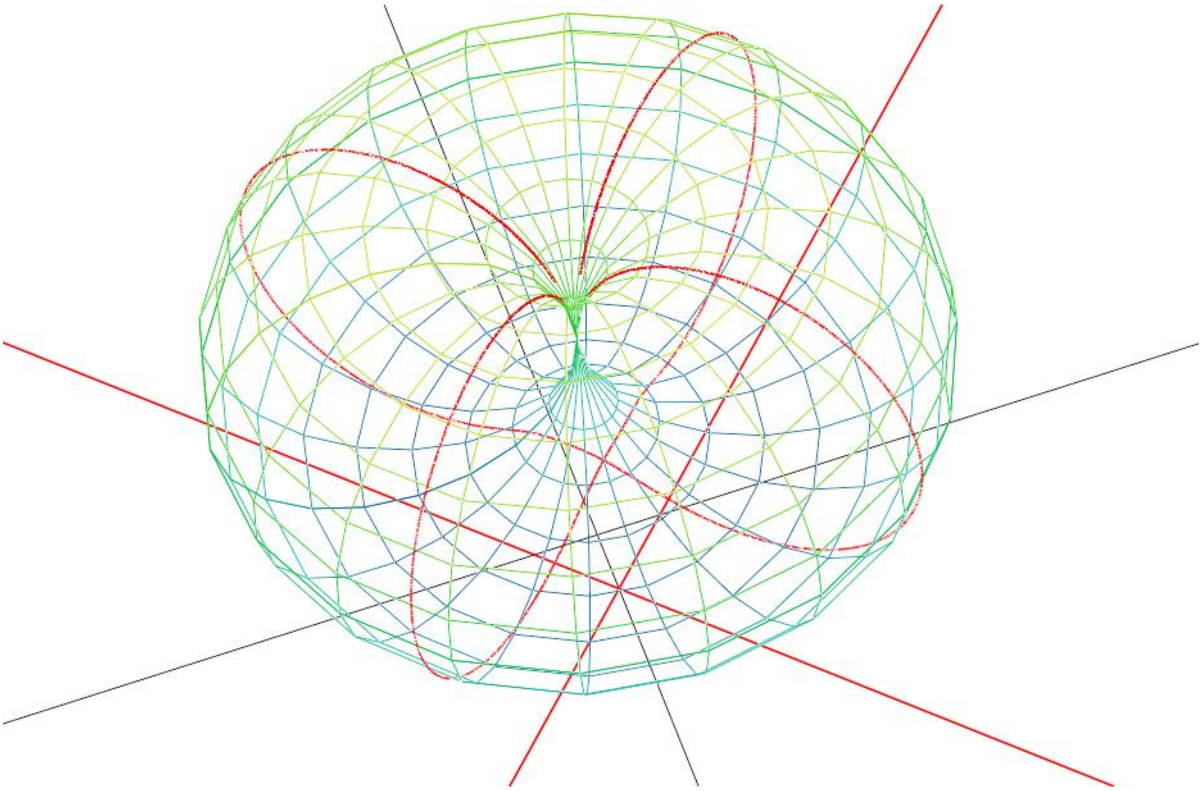
Thus, we have formulas (by Wolfgang Däumler) for mapping onto the horn torus:

$$\xi = x \cdot \frac{\frac{1}{2}\left(\sin\left(\frac{\phi}{2}\right)\right)^2}{\sqrt{x^2+y^2}}$$

$$\eta = y \cdot \frac{\frac{1}{2}\left(\sin\left(\frac{\phi}{2}\right)\right)^2}{\sqrt{x^2+y^2}}$$

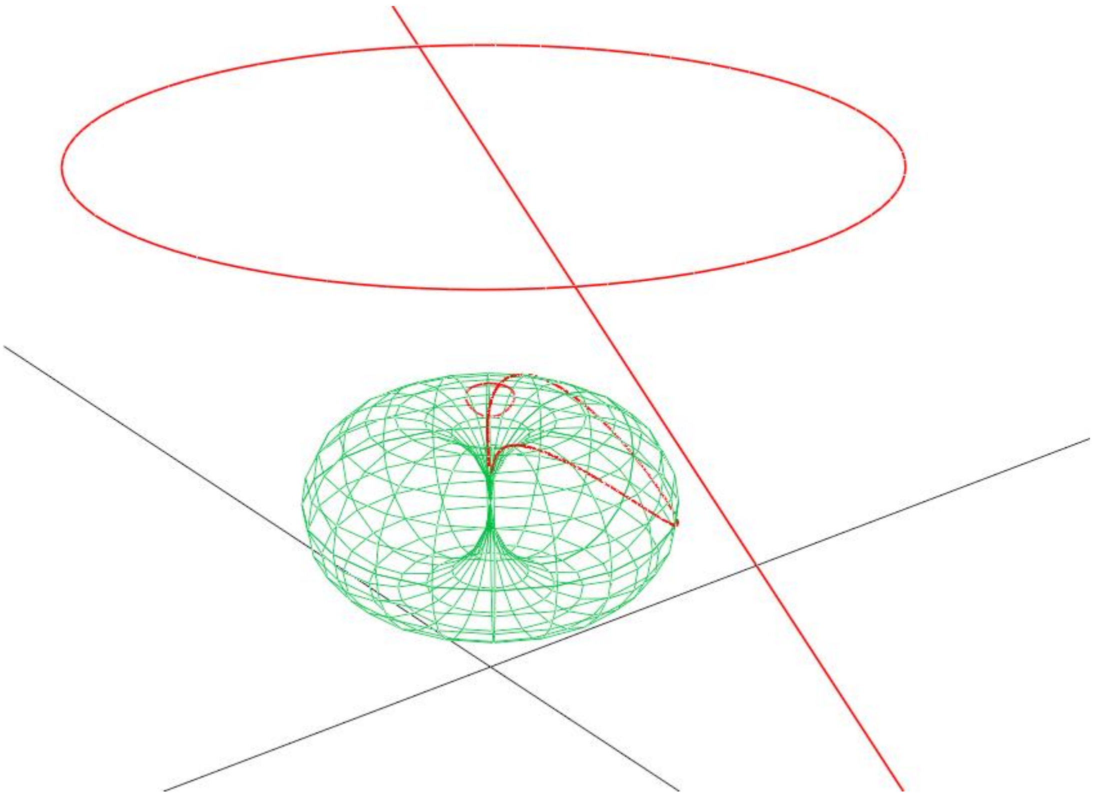
$$\zeta = \frac{1}{4}\sin\phi + \frac{1}{2}$$

The result of mapping of some lines onto the horn torus is given below.



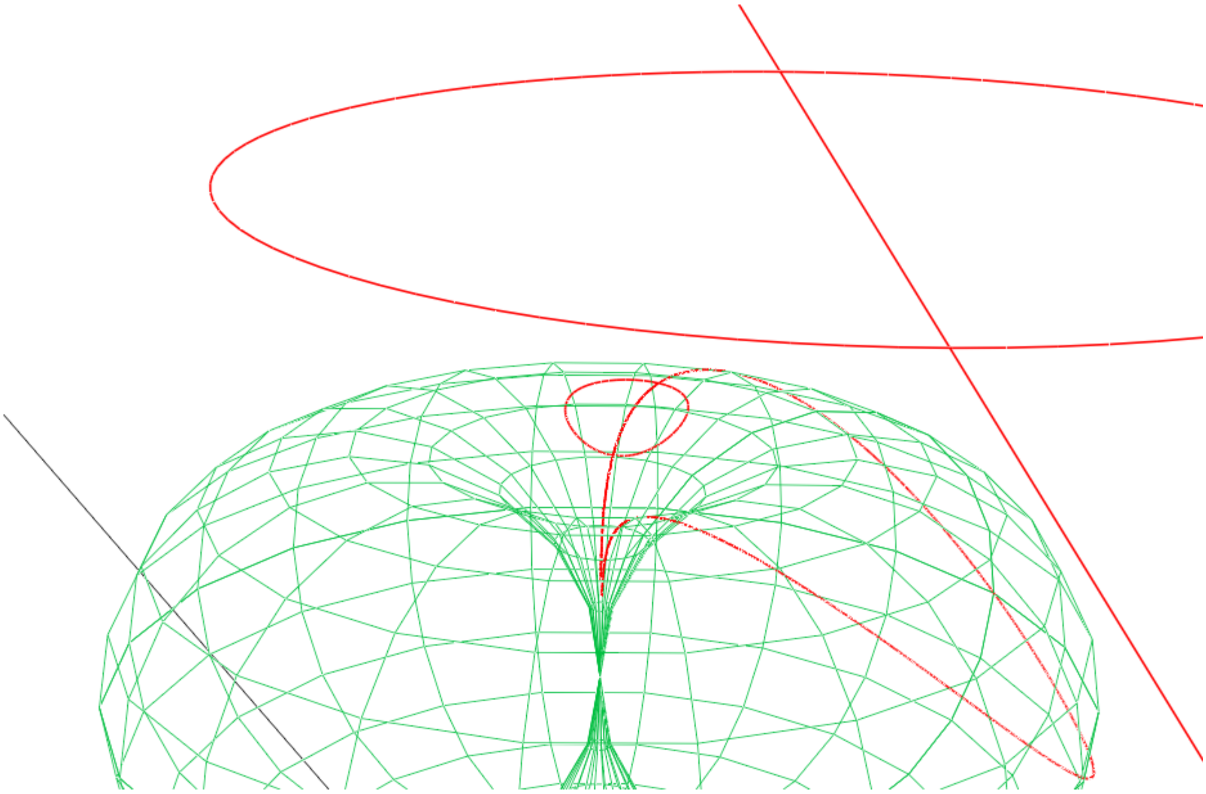
$T, S, O, O_p, Z, Z_t, Z, s_1, s_2, s_3, Ox, Oy, C_p, C_t, L_p, L_t$

Two lines passing through points $(-0.1, 0)$, $(0, 0.1)$ and $(0.1, 0)$.



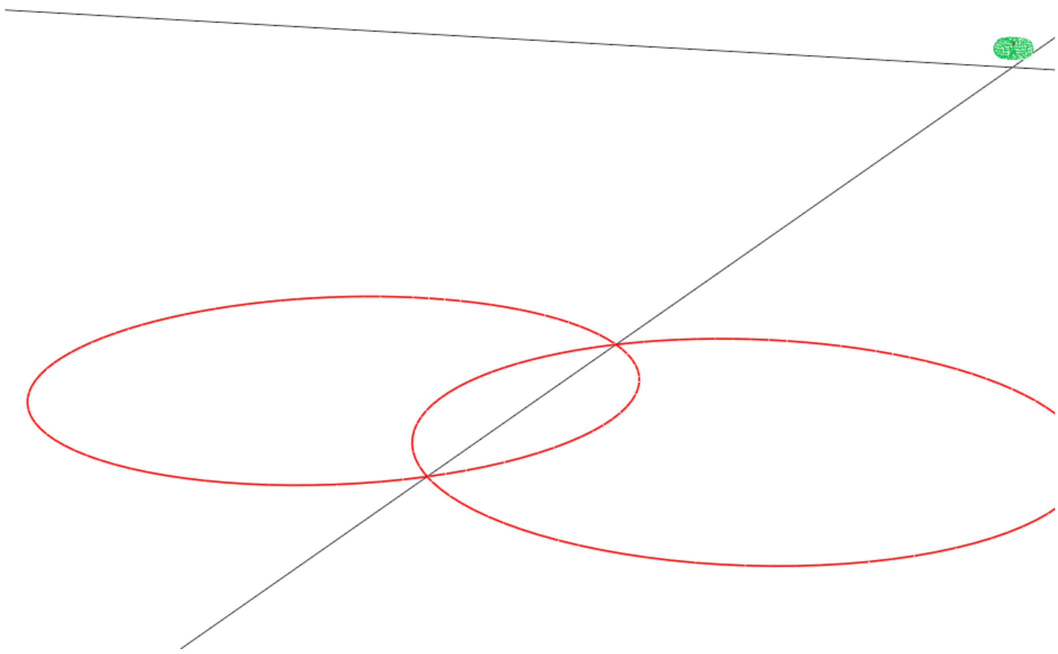
$T, S, O, O_p, Z, Z_t, Z, s_1, s_2, s_3, Ox, Oy, C_p, C_t, L_p, L_t, C_f$

A circle of radius 2 with center $(3, 4)$ and a line $y(x) = 2x - 2$.

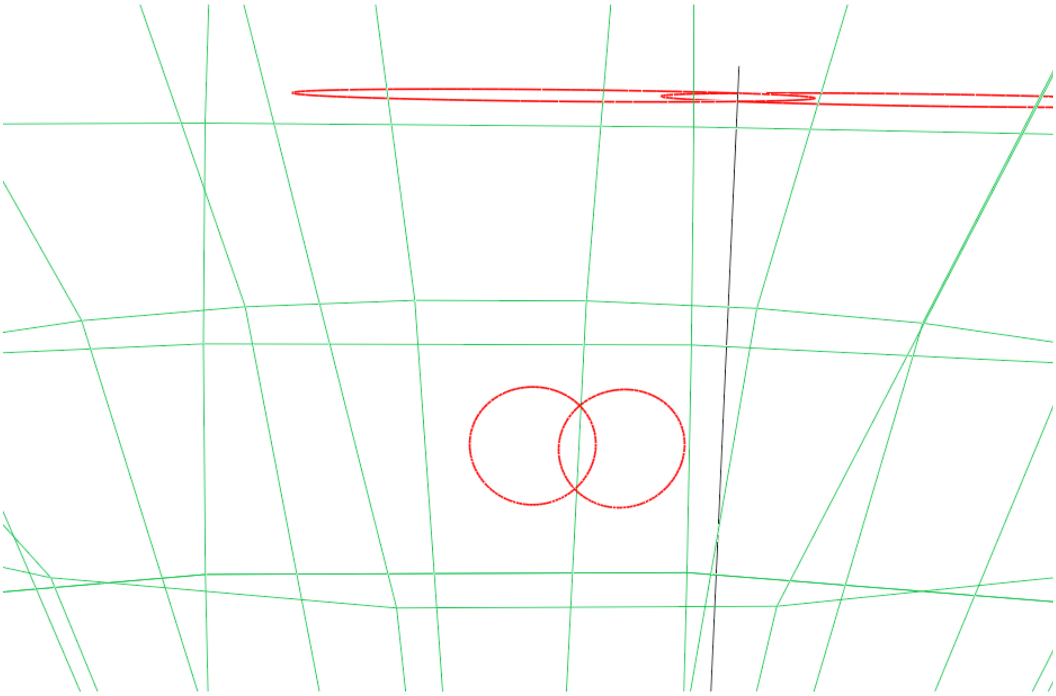


$T, S, O, O_p, Z, Z_t, Z, s_1, s_2, s_3, Ox, Oy, C_p, C_t, L_p, L_t, C_f$

The same (a circle of radius 2 with center $(3,4)$ and a line $y(x) = 2x - 2$), but under magnification.

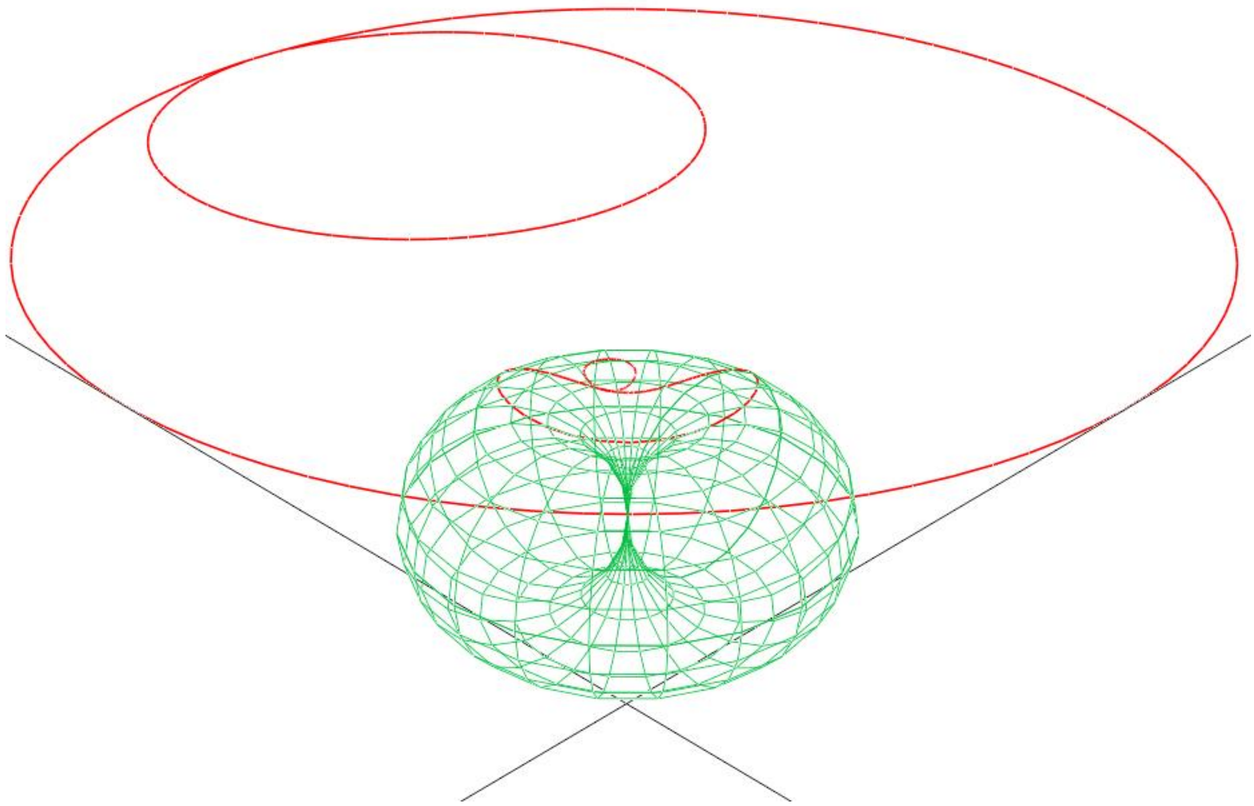


Two circles. A circle of radius 4 with center $(-\frac{\sqrt{2}}{2}, 30)$ and a circle of radius 4 with center $(\frac{\sqrt{2}}{2}, 30)$ (circles cross at right angle).



$T, S, O, O_p, Z, Z_t, Z, s_1, s_2, s_3, Ox, Oy, C_p, C_t, L_p, L_t, C_f$

The same (a circle of radius 4 with center $(-\frac{\sqrt{2}}{2}, 30)$ and a circle of radius 4 with center $(\frac{\sqrt{2}}{2}, 30)$), but under the inner part of the horn torus (two small circles at the bottom of the image are located on the surface of the horn torus).



Two circles. A circle of radius 2 with center $(2, 2)$ and a circle of radius 1 with center $(2, 3)$.

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