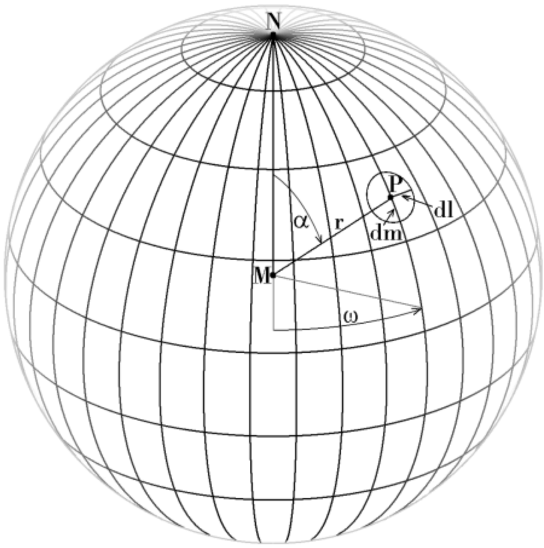
https://www.horntorus.com/manifolds/horntorusmapping.html

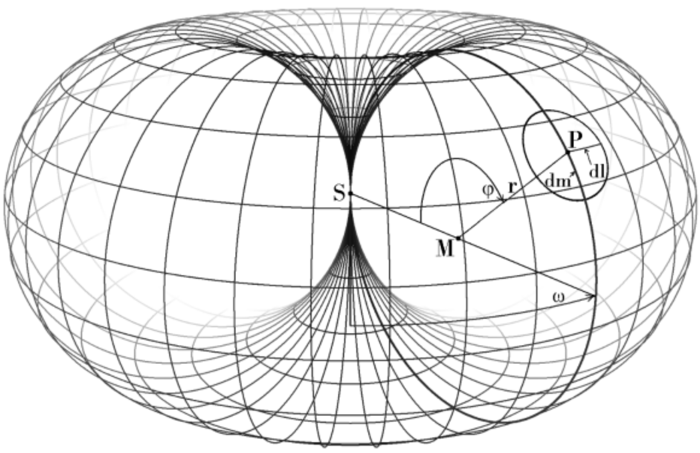
conformal mapping sphere ↔ horn torus

Due to the hope that a geometrical solution for the conformal mapping between sphere and horn torus could be found, analogue to the Riemannian stereographic projection, it first seemed necessary to nest horn torus and sphere into one another, resulting in a bit confusing drawings and derivation. But until now the search came to no result (Vyacheslav Puha's ingenious method unfortunately is not conformal, and probably no geometrical method does exist), and so we study sphere and horn torus separately.

We don't declare any appropriate stereographic projection and try to proof afterwards the conformality but we use instead the conditions of conformality to compile and establish the wanted mapping analytically:

One condition for conformal mapping is that small circles on the origin are mapped as small circles on the target surface and therefore we want to construct small circles on both surfaces that only are dependent on the position of points P on the surface, i. e. only dependent on the angles $\alpha = \angle NMP$ for the sphere resp. $\varphi = \angle SMP$ for the horn torus. Due to rotation symmetry of both figures the rotation angle ω doesn't play any role and is arbitrary, and likewise, because of mirror symmetry, α can be exchanged by $\pi - \alpha$ (what corresponds to the exchange of north and south pole of the sphere) and φ can be exchanged by $2\pi - \varphi$. Even the radii r of both figures turn out to be arbitrary and independent. Following drawing shows longitudes, spacing 10°, and latitudes, spacing 20°, P located at angles α = 50° resp. φ = 140°, ω = 35° on both figures:





We consider the small circle around any point P and state the condition that the radii dm (in direction of meridians) and dl (parallel to latitudes) have to be equal.

Lengths m of longitudes (meridians) on both figures, sphere and horn torus, are

$$m = 2\pi \cdot r$$

Lengths l of latitudes are computed differently (* see supplement for derivation):

$$l = 2\pi \cdot r \cdot \sin\alpha \text{ on the sphere and}$$
$$l = 2\pi \cdot r \cdot (1 - \cos\varphi) \text{ on horn torus. *}$$

The differentials dm - radii of the respective small circles on the longitude - are

$$dm = r \cdot d\alpha \text{ on the sphere and}$$
$$dm = r \cdot d\varphi \text{ on the horn torus.}$$

The differentials dl - radii of the respective small circles on the latitude - are

$$dl = d\omega \cdot r \cdot \sin\alpha \text{ on the sphere and}$$
$$dl = d\omega \cdot r \cdot (1 - \cos\varphi) \text{ on the horn torus.}$$

After equalling dm and dl in both figures separately and cancelling r one has

$$d\alpha = d\omega \cdot \sin\alpha \text{ for the sphere and}$$
$$d\varphi = d\omega \cdot (1 - \cos\varphi) \text{ for horn torus.}$$

By solving both equations to dω and equalling we get the differential equation

$$d\alpha/\sin\alpha = d\varphi/(1 - \cos\varphi)$$

and finally, by integration we obtain the condition for conformal mapping

$$\int(1/\sin\alpha)d\alpha = \int(1/(1 - \cos\varphi))d\varphi$$

$$\ln(|\tan(\alpha/2)|) = -\cot(\varphi/2) + C$$

sphere → horn torus:

$$\boldsymbol{\varphi = 2\cdot\text{arccot}(-\ln(|\tan(\alpha/2)|) - C)}$$

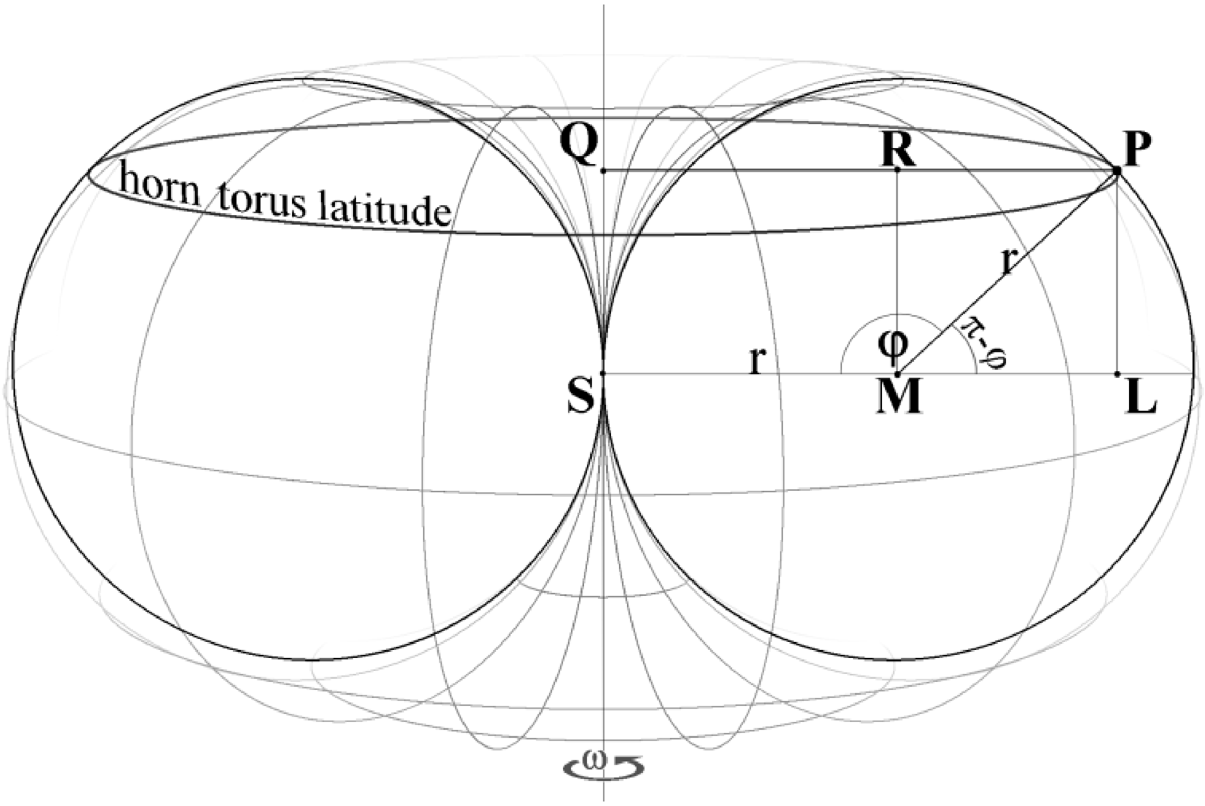
horn torus → sphere:

$$\boldsymbol{\alpha = 2\cdot\arctan(e^{(-\cot(\varphi/2) + C)})}$$

$(0 < \alpha < \pi,\ 0 < \varphi < 2\pi,\ C$ any real number$)$

C is a kind of 'zoom/diminishing factor' for the mapped figures and shifts them: case α → φ: φ moves towards 2π with increasing C > 0, towards 0 with C < 0, case φ → α: α moves towards π with increasing C > 0, towards 0 with C < 0, conformality is given for C ≠ 0 as well, i. e. there is an infinite set of solutions, but mappings are not bijective, when C ≠ 0 is the same in the inverse mapping (likewise: the Riemannian stereographic projection is a special case amongst others)

*** supplement: length of horn torus latitude**



The sketch shows details of a horn torus cross section, embedded in a slightly tilted perspective view, point P is positioned on longitude ω (rotation angle) 90° and latitude φ (torus bulge revolution angle) 135°, S is centre of horn torus, M centre of circle (half of cross section) with radius r, Q is centre of selected latitude through P and lies on main symmetry axis, auxiliary line MR is perpendicular to QP, auxiliary line PL perpendicular to diameter of horn torus cross section circle through S and M. With these points and parameters we easily can calculate length l of the latitude with radius QP:

$$
\begin{aligned}
QP &= QR + RP \\
&= SM + ML \\
&= r + r\cdot\cos(\pi - \varphi) \\
&= r - r\cdot\cos\varphi \\
&= r\cdot(1 - \cos\varphi)
\end{aligned}
$$

$$l = 2\pi\cdot QP$$

$$\mathbf{l = 2\pi\cdot r\cdot(1 - \cos\varphi)}$$

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for more on the subject visit
https://www.horntorus.com