Horn Torus Models for the Riemann Sphere From the Viewpoint of Division by Zero (draft)

Wolfgang W. Däumler, Perouse, Germany,
Hiroshi Okumura, Maebashi, Gunma, 371-0123, JAPAN
Vyacheslav Puha
Kola Science Center, Russian Academy of Sciences,
Apatity, Murmansk oblast, 184209 RUSSIA and
Saburou Saitoh
Institute of Reproducing Kernels
Kawauchi-cho, 5-1648-16, Kiryu 376-0041, JAPAN

November 27, 2018

Abstract: In this paper, we will introduce beautiful horn torus models by Puha and Däumler for the Riemann sphere in complex analysis from the viewpoint of the division by zero.

Key Words: Division by zero, division by zero calculus, singular point, $0/0 = 1/0 = z/0 = 0$, infinity, discontinuous, point at infinity, stereographic projection, Riemann sphere, horn torus, Laurent expansion, conformal mapping.

1 Division by zero calculus

The division by zero with mysterious and long history was indeed trivial and clear as in the followings:

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear all as $a/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical.

Division by zero is trivial and clear from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero.

The simple field structure containing division by zero was established by M. Yamada.

Many applications of the division by zero to Wasan geometry were given by H. Okumura.

The division by zero opens a new world since Aristotelēs-Euclid. See the references for recent related results.

As the number system containing the division by zero, the Yamada field structure is completed. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z - i}{z + i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z - (-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z - i) \cdot \frac{1}{z + i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z - i)]_{z=-i} \cdot \left[\frac{1}{z + i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n, \tag{1.5}$$

we **define** the identity, by the division by zero

$$f(a) = C_0. \tag{1.6}$$

Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.

For the correspondence (1.6) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (1.5), we can **define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n! C_n.$$

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

For the fundamental function $W = 1/z$ we did not consider any value at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [20, 21, 22]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([7, 8]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidian geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic ideas for the space and universe.

For the extended complex plane, we consider its stereographic projection mapping as the Riemann sphere and the point at infinity is realized as the north pole in the Alexsandroff's one point compactification. The Riemann sphere model gives a beautiful and complete realization of the extended complex plane through the stereographic projection mapping and the mapping has beautiful properties like isogonal (equiangular) and circle to circle correspondence (circle transformation). Therefore, the Riemann sphere is a very classical concept [1].

Now, with the division by zero we have to admit the strong discontinuity at the point at infinity, because the point at infinity is represented by zero. In [8], a formal contradiction for the classical result $1/0 = \infty$ was given and the strong discontinuity was shown in many and many examples. See the papers in the references.

On this situation, the third author discovered the mapping of the extended complex plane to a beautiful horn torus at (2018.6.4.7:22) and its inverse at (2018.6.18.22:18).

Incidentally, independently of the division by zero, Wolfgang W. Däumler has various special great ideas on horn torus as we see from his site:

Horn Torus & Physics (https://www.horntorus.com/) 'Geometry Of Everything', intellectual game to reveal engrams of dimensional thinking and proposal for a different approach to physical questions ...

Indeed, Wolfgang Däumler was presumably the first (1996) who came to the idea of the possibility of a mapping of extended complex plane onto the horn torus. He expressed this idea on his private website (http://www.dorntorus.de). He was also, apparently, the first who to point out that zero and infinity are represented by one and the same point on the horn torus model of extended complex plane.

In this paper we just introduce new horn torus models for the classical Riemann sphere from the viewpoint of the division by zero. These models seem to be important for us.

2 Horn torus model

We will consider the three circles represented by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2, \tag{2.1}$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The stereographic projection mapping from (x, y) plane to the Riemann sphere is given by

$$\xi = \frac{x}{x^2 + y^2 + 1},$$

$$\eta = \frac{y}{x^2 + y^2 + 1},$$

and

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Of course,

$$\xi^2 + \eta^2 = \zeta(1 - \zeta).$$

and

$$x = \frac{\xi}{1 - \zeta}, y = \frac{\eta}{1 - \zeta}.$$

In these formulas, we see the division by zero

$$0 = \frac{0}{0}$$

that shows the mapping from $(0, 0, 1)$ to $(0, 0)$.

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

$$\eta = \frac{2y\sqrt{x^2+y^2}}{(x^2+y^2+1)^2},$$

and

$$\zeta = \frac{(x^2+y^2-1)\sqrt{x^2+y^2}}{(x^2+y^2+1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center (0,0,1/2) and the horn torus. This is the desired point on the horn torus for the plane point.

The inversion is given by

$$x = \xi \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} \tag{2.2}$$

and

$$y = \eta \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)}. \tag{2.3}$$

In these formulas, we can see the division by zero

$$0 = \frac{0}{0}$$

that shows the mapping from $(0,0,1/2)$ to $(0,0)$.

For the properties of horn torus with physical applications, see [2].

3 Properties of horn torus model

At first, the model shows the strong symmetry of the domains $\{|z| < 1\}$ and $\{|z| > 1\}$ and they correspond to the lower part and the upper part of the horn torus, respectively. The unit circle $\{|z| = 1\}$ corresponds to the circle

$$\xi^2 + \eta^2 = \left(\frac{1}{2}\right)^2, \quad \zeta = \frac{1}{2}$$

in one to one way. Of course, the origin and the point at infinity are the same point and correspond to $(0,0,1/2)$. Furthermore, the inversion relation

$$z \longleftrightarrow \frac{1}{\overline{z}}$$

with respect to the unit circle $\{|z|=1\}$ corresponds to the relation

$$(\xi,\eta,\zeta) \longleftrightarrow (\xi,\eta,1-\zeta)$$

and similarly,

$$z \longleftrightarrow -z$$

corresponds to the relation

$$(\xi,\eta,\zeta) \longleftrightarrow (-\xi,-\eta,\zeta)$$

and

$$z \longleftrightarrow -\frac{1}{\overline{z}}$$

corresponds to the relation

$$(\xi,\eta,\zeta) \longleftrightarrow (-\xi,-\eta,1-\zeta)$$

(H.G.W. Begehr: 2018.6.18.19:20).

We can see directly the important negative properties that the mapping is not isogonal (equiangular) and infinitely small circles do not correspond to infinitely small circles, as in analytic functions.

We note that only zero and numbers a of the form $|a| = 1$ have the property : $|a|^b = |a|, b \neq 0$. Here, note that we can also consider $0^b = 0$ ([5]). The symmetry of the horn torus model agrees perfectly with this fact. Only zero and numbers a of the form $|a| = 1$ correspond to points on the plane described by equation $\zeta - 1/2 = 0$. Only zero and numbers a of the form $|a| = 1$ correspond to points whose tangent lines to the surface of the horn torus are parallel to the axis ζ.

Horn torus should be considered as simply-connected ([2], 3585). We should consider that the origin and the point at infinity (that is represented by zero) is attached as one point on the $\mathbf{R}^3$ space. Certainly, a curve through the origin and the point at infinity is mapped to a closed curve on the horn torus and the closed curve can not be shrinked to a point, however, note that the point $(0,0,1/2)$ is a boundary point on the $\mathbf{R}^3$ space.

4 Conformal mapping from the plane to the horn torus with a modified mapping

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model (2018.8.18.09):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

Our situation is invariant by rotation around ζ axis, and so we shall consider the problem on the ξ, ζ plane.

Let $N(0,0,1)$ be the north pole. Let $P'(\xi,\eta,\zeta)$ denote a point on the Riemann sphere and let $z = x+iy$ be the common point with the line NP' and $\zeta = 0$ plane (: $z = x+iy$); that is P' is the stereographic projection map of the point $z = x+iy$ onto the unit sphere.

Let $M(1/4,0,1/2)$ be the center of the circle (2.1). Let P'' be the common point of the line $SP'(S = S(0,0,1/2))$ and the circle (2.1).

Let Q' be $(0,0,\zeta)$ that is the line $Q'P'$ is parallel to the x axis. Let Q'' and M'' be the common points with the ζ axis and $\xi = 1/4$ with the parallel line to the x axis through the point P'', respectively.

Further, we set $\alpha = \angle OSP' = \angle P''IS = (1/2)\angle P''MS$ (I := I(1/2,0,1/2)). We set P for the point on the horn torus such that $\phi = \angle SMP$ and Q be the point on the ζ axis such that the line QP is parallel to the x axis.

Then, we have:

$$\overline{P'Q'} = \frac{1}{2}\sin\alpha,$$

$$\overline{P''M''} = \frac{1}{4}|\cos(2\alpha)|,$$

$$\overline{P''Q''} = \frac{1}{4}(1-\cos(2\alpha)),$$

the length of latitude through P' is

$$2\pi\overline{P'Q'} = \pi\sin\alpha,$$

and the length of latitude through P''

$$2\pi\overline{P''Q''} = \frac{\pi}{2}(1-\cos(2\alpha)) = \pi\sin^2\alpha.$$

Similarly, we have

$$2\pi\overline{QP} = \frac{\pi}{2}(1 - \cos\phi).$$

In order to become the conformal mapping from the point P' to the point P, we have the identity

$$d\alpha : d\phi = \sin\alpha : 1 - \cos\phi;$$

that is we have the differential equation

$$\frac{d\alpha}{\sin\alpha} = \frac{d\phi}{1 - \cos\phi}.$$

Note here that the radius of the circle (2.1) is half of the stereographic projection mapping circle (the Riemann sphere). We solve this differential equation as, with an integral constant C

$$\log \mid \tan\frac{\alpha}{2} \mid = -\cot\frac{\phi}{2} + C.$$

For this derivation of the differential equation, see the detail comments in the site : **conformal mapping sphere ↔ horn torus** with beautiful figures and many informations, by W. W. Däumler. In order to check his idea, we will give a complete proof analytically, in Section 6.

Using the correspondence

$$\alpha = 0 \leftrightarrow \phi = 0,$$

or

$$\alpha = \pi/2 \leftrightarrow \phi = \pi$$

or

$$\alpha = \pi \leftrightarrow \phi = 2\pi,$$

we have $C = 0$. Note that $\tan(\pi/2) = 0$, $\cot(\pi/2) = 0$ and $\log 0 = 0$ ([5]). Note also that the function $y = e^x$ takes two values 1 and 0 at $x = 0$. By solving for ϕ we have the result

$$\phi = 2\cot^{-1}(-\log \mid \tan(\alpha/2) \mid) \tag{4.1}$$

or

$$\alpha = 2\tan^{-1}(e^{(-\cot(\phi/2))}). \tag{4.2}$$

Next, note that
$$\tan\frac{\alpha}{2} = |z|$$
and
$$\alpha = 2\tan^{-1}|z|. \tag{4.3}$$
We thus have
$$\phi = 2\cot^{-1}(-\log|z|) \tag{4.4}$$
and the inverse is
$$|z| = e^{-\cot(\phi/2)}. \tag{4.5}$$

We thus obtain the complicated conformal mapping for the z plane to the horn torus by (4.4) and (4.2). The inverse conformal mapping for the horn torus to the complex z plane is given by (4.1) and (4.5).

For the integral constant C, Däumler considers the general constant C and stated that:

I don't recognize a big problem with constant C. What are the crucial points? As I stated, all mappings from sphere to horn torus and inverse with any real C are conformal, but only the mappings with $C = 0$ are bijective. Respectively with
$$\alpha = 2\tan^{-1}(P\cdot|z|)$$
and
$$|z| = \frac{\tan(\alpha/2)}{P},$$
all mappings from complex plane to sphere and inverse with real $P > 0$ are conformal, but bijective only when $P = 1$, what is the normal Riemannian stereographic projection. Main thing is to have at least one solution ($C = 0$) in this topic, and we can keep other constants, C not equal 0 and P not equal 1, for special cases in different context.

For this very interesting topics, see his site.

We can represent the direct Däumler mapping from the z plane onto the horn torus as follows (V. Puha: 2018.8.28.22:31):
$$\xi = \frac{x\cdot(1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}}, \tag{4.6}$$
$$\eta = \frac{y\cdot(1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}}, \tag{4.7}$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}. \tag{4.8}$$

Indeed, at first, we have

$$SP := L = 2\cdot\frac{1}{4}\sin\frac{\phi}{2} = \frac{1}{2}\sin\frac{\phi}{2}, \tag{4.9}$$

$$\sqrt{\xi^2+\eta^2} = L\cos\left(\frac{\pi}{2}-\frac{\phi}{2}\right) = L\sin\frac{\phi}{2},$$

and

$$\sqrt{\xi^2+\eta^2} = \frac{1}{2}\sin^2\frac{\phi}{2}.$$

From the simple relations

$$\xi = \frac{x\sqrt{\xi^2+\eta^2}}{\sqrt{x^2+y^2}}, \eta = \frac{y\sqrt{\xi^2+\eta^2}}{\sqrt{x^2+y^2}}, \tag{4.10}$$

and

$$\zeta = -L\sin\left(\frac{\pi}{2}-\frac{\phi}{2}\right) + \frac{1}{2},$$

we have the desired representations.

We will give the inversion formula of the Däumler mapping. From (4.10) we have

$$x = \frac{\xi\sqrt{x^2+y^2}}{\sqrt{\xi^2+\eta^2}}, y = \frac{\eta\sqrt{x^2+y^2}}{\sqrt{\xi^2+\eta^2}}. \tag{4.11}$$

Hence, it is enough to represent of $\sqrt{x^2+y^2}$ in terms of ξ, η, ζ on the horn torus. From (4.5), (4.9) and

$$T = \sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2},$$

we have the inversion formula from the horn torus to the x, y plane.

$$x = \frac{\xi}{\sqrt{\xi^2+\eta^2}}\exp\pm\left\{\frac{\sqrt{\zeta-(\xi^2+\eta^2+\zeta^2)}}{\sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2}}\right\} \tag{4.12}$$

and

$$y = \frac{\eta}{\sqrt{\xi^2+\eta^2}}\exp\pm\left\{\frac{\sqrt{\zeta-(\xi^2+\eta^2+\zeta^2)}}{\sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2}}\right\}. \tag{4.13}$$

.

5 Properties of the Däumler conformal mapping

The Däumler conformal mapping stated in Section 4 is very complicated, however, has very beautiful properties. We will see its elementary properties.

The circle $|z| = r$ is mapped to the circle:

$$\xi^2 + \eta^2 = \frac{1}{4}\left\{\sin\frac{\phi}{2}\right\}^4, \quad \zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}$$

with

$$\frac{\phi}{2} = \cot^{-1}(-\log r).$$

In particular, note that the unit circle $r = 1$ is mapped to the circle

$$\xi^2 + \eta^2 = \left(\frac{1}{2}\right)^2, \quad \zeta = \frac{1}{2}.$$

Here, note also that $\phi = \pi$, by using the division by zero calculus, from

$$\frac{1}{\tan(\phi/2)} = 0.$$

We have the relation

$$\frac{\eta}{\xi} = \frac{y}{x},$$

but for $y = mx$

$$\zeta = -\frac{1}{4}\sin\left\{2\cot^{-1}\left(-\frac{1}{2}(\log x^2 + \log(1 + m^2))\right)\right\}.$$

Furthermore, the inversion relation

$$z \longleftrightarrow \frac{1}{\overline{z}}$$

with respect to the unit circle $\{|z| = 1\}$ corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (\xi, \eta, 1 - \zeta)$$

and similarly,

$$z \longleftrightarrow -z$$

corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (-\xi, -\eta, \zeta)$$

and

$$z \longleftrightarrow -\frac{1}{\overline{z}}$$

corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (-\xi, -\eta, 1-\zeta).$$

Of course, the conformal mapping of Däumler is important, however, its mapping is very involved and the difference with the Puha mapping in Section 3 is just the shift on the circle of longitude and the Puha mapping is very simple. Furthermore the Puha mapping is clear in the geometrical correspondence. Therefore, we will be able to enjoy the Puha mapping for the horn torus model.

6 Proof of conformal mapping

In order to confirm the conformal mapping of Däumler mapping and at the same time, in order to see its analytical structure, we will examine it. First, we calcululate the first order derivatives.

$$\frac{\partial \xi}{\partial x} = \frac{8y^2 - 8x^2 \log[x^2+y^2] + 2y^2 \log[x^2+y^2]^2}{(x^2+y^2)^(3/2)(4+\log[x^2+y^2]^2)^2},$$

$$\frac{\partial \xi}{\partial y} = -\frac{(2xy(2+\log[x^2+y^2])^2}{(x^2+y^2)^(3/2)(4+\log[x^2+y^2]^2)^2)},$$

$$\frac{\partial \eta}{\partial x} = -\frac{2xy(2+\log[x^2+y^2])^2}{(x^2+y^2)^(3/2)(4+\log[x^2+y^2]^2)^2)},$$

$$\frac{\partial \eta}{\partial y} = \frac{\partial \zeta}{\partial x} = -\frac{2x(-4+\log[x^2+y^2]^2)}{(x^2+y^2)(4+\log[x^2+y^2]^2)^2)},$$

and

$$\frac{\partial \zeta}{\partial y} = -\frac{2y(-4+\log[x^2+y^2]^2)}{(x^2+y^2)(4+\log[x^2+y^2]^2)^2)}.$$

Next, we wish to have the relation between

$$(d\sigma)^2 = (d\xi)^2 + (d\eta)^2 + (d\zeta)^2$$

and

$$(ds)^2 = (dx)^2 + (dy)^2.$$

From

$$d\xi = \frac{2(-xydy(2+\log[x^2+y^2])^2 + dx(4y^2 - 4x^2\log[x^2+y^2] + y^2\log[x^2+y^2]^2))}{(x^2+y^2)^(3/2)(4+\log[x^2+y^2]^2)^2},$$

$$d\eta = \frac{2(-xydx(2+\log[x^2+y^2])^2 + dy(4x^2 - 4y^2\log[x^2+y^2] + x^2\log[x^2+y^2]^2))}{(x^2+y^2)^(3/2)(4+\log[x^2+y^2]^2)^2},$$

$$d\zeta = -\frac{(2(xdx+ydy)(-4+\log[x^2+y^2]^2)}{(x^2+y^2)(4+\log[x^2+y^2]^2)^2)},$$

we obtain the beautiful identity

$$(d\sigma)^2 = \frac{4(ds)^2}{(x^2+y^2)(4+\log[x^2+y^2]^2)^2}. \tag{6.1}$$

The next and final crucial point is the relation:

$$\frac{dx}{ds}, \frac{dy}{ds}$$

and

$$\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}, \frac{d\zeta}{d\sigma}.$$

This may be done directly by division by $d\sigma$ in (6.1). Indeed, we have:

$$\frac{d\xi}{d\sigma} = (dx(y^2 - (4x^2\log[x^2+y^2])/(4+\log[x^2+y^2]^2)))/(ds(x^2+y^2)) \tag{6.2}$$

$$+(dy(-xy - (4xy\log[x^2+y^2])/(4+\log[x^2+y^2]^2)))/(ds(x^2+y^2))$$

$$= \frac{dx\left(y^2 - \frac{4x^2\log\left[x^2+y^2\right]}{4+\log[x^2+y^2]^2}\right)}{ds\,(x^2+y^2)} + \frac{dy\left(-xy - \frac{4xy\log\left[x^2+y^2\right]}{4+\log[x^2+y^2]^2}\right)}{ds\,(x^2+y^2)},$$

$$\frac{d\eta}{d\sigma} = (dx(-xy - (4xy\log[x^2+y^2])/(4+\log[x^2+y^2]))))/(ds(x^2+y^2)) \quad (6.3)$$

$$+(dy(x^2 - (4y^2\log[x^2+y^2])/(4+\log[x^2+y^2]^2))))/(ds(x^2+y^2)),$$

and

$$\frac{d\zeta}{d\sigma} = -((xdx(-4+\log[x^2+y^2]^2))/(ds\sqrt{x^2+y^2}(4+\log[x^2+y^2]^2))) \quad (6.4)$$

$$-(ydy(-4+\log[x^2+y^2]^2))/(ds\sqrt{x^2+y^2}(4+\log[x^2+y^2]^2)).$$

On a point $P_0(\xi_0, \eta_0, \zeta_0)$ on the horn torus we consider two smooth curves passing the point

$$f_j(\xi, \eta, \zeta) = 0, \quad j = 1, 2.$$

At the point P_0, we denote the values of $\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}, \frac{d\zeta}{d\sigma}$ by λ_j, μ_j, ν_j, respectively. Then, for the angle Φ made by the curves at the point P_0 we have

$$\cos\Phi = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2. \quad (6.5)$$

The corresponding relations on the x, y plane are as follows:

For the corresponding curves on the x, y plane

$$g_j(x, y) = 0, \quad j = 1, 2, \quad (6.6)$$

at the corresponding point $Q_0(x_0, y_0)$, we denote the values of $\frac{dx}{ds}, \frac{dy}{ds}$ by α_j, β_j, respectively. Then, for the angle ϕ of the curves at the point $Q_0(x_0, y_0)$ we have

$$\cos\phi = \alpha_1\beta_1 + \alpha_2\beta_2. \quad (6.7)$$

We wish to prove that (6.5) = (6.7), by formal calculation.

Note that from (6.2), we have for $(x, y) = (x_0, y_0)$, here, for simplicity we shall use (x, y) at Q_0

$$\lambda_j = \frac{\alpha_j\left(y^2 - \frac{4x^2\log\left[x^2+y^2\right]}{4+\log[x^2+y^2]^2}\right)}{(x^2+y^2)} + \frac{\beta_j\left(-xy - \frac{4xy\log\left[x^2+y^2\right]}{4+\log[x^2+y^2]^2}\right)}{(x^2+y^2)}. \quad (6.8)$$

Similarly, from (6.3),

$$\mu_j = (\alpha_j(-xy - (4xy\log[x^2+y^2])/(4+\log[x^2+y^2]))))/((x^2+y^2)) \quad (6.9)$$

$$+(\beta_j(x^2-(4y^2\log[x^2+y^2])/(4+\log[x^2+y^2]^2)))/((x^2+y^2)),$$

and from (6.4),

$$\nu_j=-((x\alpha_j(-4+\log[x^2+y^2]^2))/(\sqrt{x^2+y^2}(4+\log[x^2+y^2]^2))) \quad (6.10)$$
$$-(y\beta_j(-4+\log[x^2+y^2]^2))/(\sqrt{x^2+y^2}(4+\log[x^2+y^2]^2)).$$

When we insert these in the right hand side of (6.5), we obtain the desired identity. In this section, for these formal calculations, we used MATHEMATICA.

7 Acknowledgements

The fourth author wishes to express his deep thanks Professors H.G.W. Begehr, Tao Qian and Masakazu Shiba for their exciting communications.

References

[1] L. V. Ahlfors, Complex Analysis, McGraw-Hill Book Company, 1966.

[2] M. Beleggia, M. De. Graef and Y. T. Millev, Magnetostatics of the uniformly polarized torus, Proc. R. So. A(2009), **465**, 3581–3604.

[3] M. Kuroda, H. Michiwaki, S. Saitoh, and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math. **27** (2014), no 2, pp. 191-198, DOI: 10.12732/ijam.v27i2.9.

[4] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, Advances in Linear Algebra & Matrix Theory, **6**(2016), 51-58 Published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt http://dx.doi.org/10.4236/alamt.2016.62007.

[5] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log\infty = 0$ and applications. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics & Statistics. **230** (2018), 293-305.

[6] H. Michiwaki, S. Saitoh and M.Yamada, Reality of the division by zero $z/0 = 0$. IJAPM International J. of Applied Physics and Math. **6**(2015), 1–8. http://www.ijapm.org/show-63-504-1.html

[7] H. Michiwaki, H. Okumura and S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, International Journal of Mathematics and Computation, **2**8(2017); Issue 1, 1-16.

[8] H. Okumura, S. Saitoh and T. Matsuura, Relations of 0 and ∞, Journal of Technology and Social Science (JTSS), **1**(2017), 70-77.

[9] H. Okumura and S. Saitoh, The Descartes circles theorem and division by zero calculus. https://arxiv.org/abs/1711.04961 (2017.11.14).

[10] H. Okumura, Wasan geometry with the division by 0. https://arxiv.org/abs/1711.06947 International Journal of Geometry.

[11] H. Okumura and S. Saitoh, Harmonic Mean and Division by Zero, Dedicated to Professor Josip Pečarić on the occasion of his 70th birthday, Forum Geometricorum, **18** (2018), 155—159.

[12] H. Okumura and S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by H. Okumura and M. Watanabe, Forum Geometricorum, **18**(2018), 97-100.

[13] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry. GLOBAL JOURNAL OF ADVANCED RESEARCH ON CLASSICAL AND MODERN GEOMETRIES" (GJARCMG), **7**(2018), 2, 44–49.

[14] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics & Statistics. **230** (2018), 399-418.

[15] S. Saitoh, Generalized inversions of Hadamard and tensor products for matrices, Advances in Linear Algebra & Matrix Theory. **4** (2014), no. 2, 87–95. http://www.scirp.org/journal/ALAMT/

[16] S. Saitoh, A reproducing kernel theory with some general applications, Qian,T./Rodino,L.(eds.): Mathematical Analysis, Probability and Applications - Plenary Lectures: Isaac 2015, Macau, China, Springer Proceedings in Mathematics and Statistics, **177**(2016), 151-182.

[17] S. Saitoh, Mysterious Properties of the Point at Infinity, arXiv:1712.09467 [math.GM](2017.12.17).

[18] S. Saitoh, Division by zero calculus (211 pages): http//okmr.yamatoblog.net/

[19] S.-E. Takahasi, M. Tsukada and Y. Kobayashi, Classification of continuous fractional binary operations on the real and complex fields, Tokyo Journal of Mathematics, **38**(2015), no. 2, 369-380.

[20] https://philosophy.kent.edu/OPA2/sites/default/files/012001.pdf

[21] http://publish.uwo.ca/ jbell/The 20Continuous.pdf

[22] http://www.mathpages.com/home/kmath526/kmath526.htm

E-mail: artmetic@gmx.de, okumura.hirosi@palette.plala.or.jp, puhanot@gmail.com, saburou.saitoh@gmail.com